ISSN 2229-5518

A Class of New Block Generalized Adams Implicit Runge-Kutta Collocation Methods

[1]Kumleng G.M., [2]Chollom J. P, [3]Omagwu S.

Abstract— In this paper, the reformulation of the block generalized Adams methods into block generalized Adams implicit Rung-Kutta methods for step numbers k =3, 4, 5, is considered. This is because of the usefulness of block implicit Runge-Kutta methods for the solution of stiff ordinary differential equations. The new methods proposed in this paper turn out to be A-stable and possess the stability properties of the Runge-Kutta methods and have implicit structure for accurate and efficient implementation. Numerical examples obtained demonstrate the accuracy and efficiency of the new block methods.

Index Terms— A-stable, Generalized Adams methods, Rung-Kutta methods and Stiff Ordinary Differential Equations.

1 Introduction

In an earlier work, Chollom and Kumleng (2012) constructed and implemented a class of block Adams Moulton implicit Runge – Kutta(Bhamirk) methods for stiff differential equations. The Bhamirk methods were shown to be A-stable and results obtained converged to the exact solution. Yakubu (2004), constructed a one-step block implicit Runge – Kutta collocation methods for solving stiff ODEs, the schemes were obtained based on collocation at Gaussian points and were shown to possess large region of absolute stability. Other authors such as Yahaya and Adegboye (2007, 2013) have constructed implicit block Runge -Kutta methods based on the Quade's type for the solution of stiff ODEs. The implicit Runge-Kutta methods have been shown to be very effective in the treatment of stiff initial value problems as can be seen in the work of Axelson (1969), Chipman (1971). The implicit Runge-Kutta methods were fully developed based on the Gaussian, Lobatto and Radauquadritures by Butcher and Wanner (1996). Gonzalez-Pinto et al (1997) developed a single –Newton schemes for the solution of the stage equations of some implicit Runge-kutta methods such as Gauss, RadauIIA and Lobatto IIIA with four implicit stages.

In this paper, the reformulation of the block generalized Adams methods as Block generalized implicit Runge-Kutta (BGAIRK) methods is considered for the solution of initial value problems of the form

$$y' = f(x, y), \quad y(x_0) = y_0, \quad x \in [a, b], \quad y \in R \tag{1}$$

and let y_n, be an approximation to $y(x_n)$, $x_n = x_0 + nh,\ n = 1,2,3....$ An implicit Runge-Kutta method has the form $y_{n+1} = y_n + h\sum_{i=1}^{s} b_i k_i$ where

$$k_i = hf(t_n + c_i h, y_n + \sum_{j=1}^{s} a_{ij} k_j), \quad i = 1,..., s \tag{2}$$

The Butcher tableau for the method is given as

$$\begin{array}{c|cccc} c_1 & a_{11} & a_{12} & \cdots & a_{1s} \\ c_2 & a_{21} & a_{22} & \cdots & a_{2s} \\ \vdots & & & \ddots & \\ c_s & a_{s1} & a_{s2} & \cdots & a_{ss} \\ \hline & b_1 & b_2 & \cdots & b_s \end{array}$$

For the implicit Runge –Kutta methods, at every step a system of algebraic equations with s components has to be solved. This increases the computational cost considerably.

The New Method

The construction of block generalized Adams methods for step numbers 2,3,…10 have been considered in Kumleng (2013)using the continuous finite difference approximation and the interpolation and collocation criteria described by Lie and Norsett (1989) and Onumanyi et.al(1994). These block methods were shown to have good stability properties making them useful in the solution of stiff ODEs. To further improve on the good stability properties of these block methods, the reformulation of the methods into block implicit Runge-Kutta methods for step numbers k =3,4,5 is considered in this paper. The reformulation of the block generalized Adams methodis considered here because of the advantages that the implicit Runge-Kutta methods have over the conventional linear multistep methods. The block generalized Adams methods considered by Kumleng (2013) for step numbers k =3, 4, 5 are as follows;

Block generalized Adams method for step number 3 with uniform order 4

$$
\begin{aligned}
y_{n+1}-y_{n} &= \tfrac{h}{24}(9f_{n}+19f_{n+1}-5f_{n+2}+f_{n+3})\\
y_{n+2}-y_{n+1} &= \tfrac{h}{24}(-f_{n}+13f_{n+1}+13f_{n+2}-f_{n+3})\\
y_{n+3}-y_{n+1} &= \tfrac{h}{3}(f_{n+1}+4f_{n+2}+f_{n+3})
\end{aligned}
\tag{3}
$$

Block generalized Adams method for step number 4 with uniform order 5

$$
\begin{aligned}
y_{n+1}-y_{n} &= \tfrac{h}{720}(251f_{n}+646f_{n+1}-264f_{n+2}+106f_{n+3}-19f_{n+4})\\
y_{n+2}-y_{n+1} &= \tfrac{h}{720}(-19f_{n}+346f_{n+1}+456f_{n+2}-74f_{n+3}+11f_{n+4})\\
y_{n+3}-y_{n+1} &= \tfrac{h}{90}(-f_{n}+34f_{n+1}+114f_{n+2}+34f_{n+3}-f_{n+4})\\
y_{n+4}-y_{n+1} &= \tfrac{h}{80}(-3f_{n}+42f_{n+1}+72f_{n+2}+102f_{n+3}+27f_{n+4})
\end{aligned}
\tag{4}
$$

Block generalized Adams method for step number 5 with uniform order 6

$$
\begin{aligned}
y_{n+1}-y_{n+2} &= \tfrac{h}{1440}(27f_{n}-637f_{n+1}-1022f_{n+2}+258f_{n+3}-77f_{n+4}+11f_{n+5})\\
y_{n+2}-y_{n} &= \tfrac{h}{90}(28f_{n}+129f_{n+1}+14f_{n+2}+14f_{n+3}-6f_{n+4}+f_{n+5})\\
y_{n+3}-y_{n+2} &= \tfrac{h}{1440}(11f_{n}-93f_{n+1}+802f_{n+2}+802f_{n+3}-93f_{n+4}+11f_{n+5})\\
y_{n+4}-y_{n+2} &= \tfrac{h}{90}(-f_{n+1}+34f_{n+2}+114f_{n+3}+34f_{n+4}-f_{n+5})\\
y_{n+5}-y_{n+2} &= \tfrac{h}{160}(3f_{n}-21f_{n+1}+114f_{n+2}+114f_{n+3}+219f_{n+4}+51f_{n+5})
\end{aligned}
\tag{5}
$$

respectively.

2 **Derivation of BGAIRK Methodk=3**

The block GAMs (3) are reformulated into the BGAIRK Method of order four by solving them simultaneously to arrive at

$$
\begin{aligned}
y_{n+1}-y_{n} &= \tfrac{h}{24}(9k_{1}+19k_{2}-5k_{3}+k_{4})\\
y_{n+2}-y_{n} &= \tfrac{h}{3}(k_{1}+4k_{2}+k_{3})\\
y_{n+3}-y_{n} &= \tfrac{h}{8}(3k_{1}+9k_{2}+9k_{3}+3k_{4})
\end{aligned}
\tag{6}
$$

Its GAIRK method which is a four – stage implicit Runge – Kutta collocation family of uniform order four is given as

$$
\begin{aligned}
k_{1} &= f(x_{n},y_{n})\\
k_{2} &= f(x_{n}+h,y_{n+1})\\
k_{3} &= f(x_{n}+2h,y_{n+2})\\
k_{4} &= f(x_{n}+3h,y_{n+3})
\end{aligned}
\tag{7}
$$

ISSN 2229-5518

The characteristic coefficients of the BGAIRK method are displayed in form of the Butcher tableau as follows:

$$
\begin{array}{c|cccc}
0 & 0 & 0 & 0 & 0 \\
1 & \frac{9}{24} & \frac{19}{24} & -\frac{5}{24} & 1 \\
2 & \frac{1}{3} & \frac{4}{3} & \frac{1}{3} & 0 \\
3 & \frac{3}{8} & \frac{9}{8} & \frac{9}{8} & \frac{3}{8} \\
\hline
 & \frac{9}{24} & \frac{19}{24} & -\frac{5}{24} & 1
\end{array}
$$

Derivation of BGAIRK Method k=4

The block GAMs (4) are also reformulated into the BGAIRK Method of order five by solving them simultaneously to arrive at

$$
\begin{aligned}
y_{n+1} - y_n &= \tfrac{h}{720}(251k_1 + 646k_2 - 264k_3 + 106k_4 - 19k_5) \\
y_{n+2} - y_n &= \tfrac{h}{90}(29k_1 + 124k_2 + 24k_3 + 4k_4 - k_5) \\
y_{n+3} - y_n &= \tfrac{h}{80}(27k_1 + 102k_2 + 72k_3 + 42k_4 - 3k_5) \\
y_{n+4} - y_n &= \tfrac{h}{45}(14k_1 + 64k_2 + 24k_3 + 64k_4 + 14k_5)
\end{aligned}
\tag{8}
$$

Its GAIRK method which is a five– stage implicit Runge – Kutta collocation family of uniform order five is given as

$$
\begin{aligned}
k_1 &= f(x_n, y_n) \\
k_2 &= f(x_n + h, y_{n+1}) \\
k_3 &= f(x_n + 2h, y_{n+2}) \\
k_4 &= f(x_n + 3h, y_{n+3}) \\
k_5 &= f(x_n + 4h, y_{n+4})
\end{aligned}
\tag{9}
$$

The characteristic coefficients of the BGAIRK method are displayed in form of the Butcher tableau as follows:

$$
\begin{array}{c|ccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
1 & \frac{251}{720} & \frac{646}{720} & -\frac{264}{720} & \frac{106}{720} & -\frac{19}{720} \\
2 & \frac{29}{90} & \frac{124}{90} & \frac{24}{90} & \frac{4}{90} & -\frac{1}{90} \\
3 & \frac{27}{80} & \frac{102}{80} & \frac{72}{80} & \frac{42}{80} & -\frac{3}{80} \\
4 & \frac{14}{45} & \frac{64}{45} & \frac{24}{45} & \frac{64}{45} & \frac{14}{45} \\
\hline
 & \frac{251}{720} & \frac{646}{720} & -\frac{264}{720} & \frac{106}{720} & -\frac{19}{720}
\end{array}
$$

ISSN 2229-5518

Derivation of BGAIRK Method k=5

The block GAMs (5) are reformulated into the BGAIRK Method of order six by solving them simultaneously to arrive at

$$\begin{aligned}
y_{n+1}-y_n &= \tfrac{h}{1440}(475k_1+1427k_2-798k_3+482k_4-173k_5+27k_6)\\
y_{n+2}-y_n &= \tfrac{h}{90}(28k_1+129k_2+14k_3+14k_4-6k_5+k_6)\\
y_{n+3}-y_n &= \tfrac{h}{160}(51k_1+219k_2+114k_3+114k_4-21k_5+3k_6)\\
y_{n+4}-y_n &= \tfrac{h}{45}(14k_1+64k_2+24k_3+64k_4+14k_5)\\
y_{n+5}-y_n &= \tfrac{h}{288}(95k_1+375k_2+250k_3+250k_4+375k_5+95k_6)
\end{aligned} \qquad (10)$$

The GAIRK method for (10) which is a six– stage implicit Runge – Kutta collocation family of uniform order six is given as

$$\begin{aligned}
k_1 &= f(x_n, y_n)\\
k_2 &= f(x_n+h, y_{n+1})\\
k_3 &= f(x_n+2h, y_{n+2})\\
k_4 &= f(x_n+3h, y_{n+3})\\
k_5 &= f(x_n+4h, y_{n+4})\\
k_6 &= f(x_n+5h, y_{n+5})
\end{aligned} \qquad (11)$$

The characteristic coefficients of the BGAIRK method are displayed in form of the Butcher tableau as follows:

0	0	0	0	0	0	0
1	$\frac{475}{1440}$	$\frac{1427}{1440}$	$-\frac{798}{1440}$	$\frac{482}{1440}$	$-\frac{173}{1440}$	$\frac{27}{1440}$
2	$\frac{28}{90}$	$\frac{129}{90}$	$\frac{14}{90}$	$\frac{14}{90}$	$-\frac{6}{90}$	$\frac{1}{90}$
3	$\frac{51}{160}$	$\frac{219}{160}$	$\frac{114}{160}$	$\frac{114}{160}$	$-\frac{21}{160}$	$\frac{3}{160}$
4	$\frac{14}{45}$	$\frac{64}{45}$	$\frac{24}{45}$	$\frac{64}{45}$	$\frac{14}{45}$	0
5	$\frac{95}{288}$	$\frac{375}{288}$	$\frac{250}{288}$	$\frac{250}{288}$	$\frac{375}{288}$	$\frac{95}{288}$
	$\frac{475}{1440}$	$\frac{1427}{1440}$	$-\frac{798}{1440}$	$\frac{482}{1440}$	$-\frac{173}{1440}$	$\frac{27}{1440}$

Absolute Stability Regions of the GAIRK Methods

In this section, the plots of the region of absolute stability of the newly constructed GAIRK methods for k=s =3, 4, 5 are considered. This is done by reformulating the GAIRK methods as general linear methods introduced by Burrage and Butcher (1980), Butcher and Wanner(1996) and further considered by Chollom (2007). The regions of absolute stability of the GAIRK methods are as shown in fig.

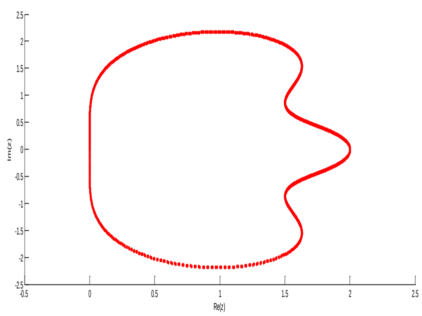

Fig1. Absolute Stability Regions of the GAIRK method k=3

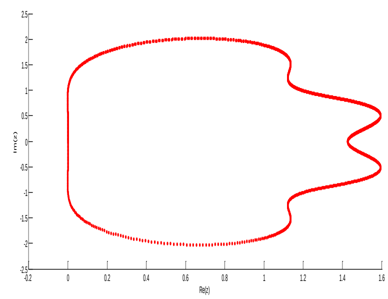

Fig2. Absolute Stability Regions of the GAIRKmethod k=4

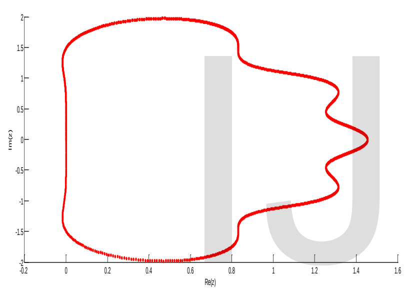

Fig3. Absolute Stability Regions of the GAIRK method k=5

The regions in Fig1 and 2 shows that the GAIRK methods for k=3 and 4 are A-stable since their regions of absolute stability contains the whole of the left-hand half plane while Fig 3 reveals that the GAIRK method for step number 5 is A(a)-Stable.

ISSN 2229-5518

Numerical Experiments

In order to ascertain the accuracy and efficiency of the GAIRK methods, numerical examples are considered. Results obtained are compared with LobattoIIIA methods of orders 4 and 6 and the RadauIIAmethod of order 5because of their possession of excellent stability and convergence properties for stiff problems (see Dekker and Verwer (1984), Butcher (1987), Hairer and Wanner (1996)).In all the problems solved, h =0.1 and $0 \le x \le 1$.

Example1. See Chollom and Kumleng (2013)

$$y' = 100(\sin x - y), \;\; y_0 = 0$$

$$y(x) = \frac{(\sin x - 0{,}01\cos x + 0.01e^{-100x})}{1.001}$$

Example2. See Yakubu*et.al* (2011)

$$y' = -100(y - x^3) + 3x^2, \;\; y(0) = 1,$$

$$y(x) = x^3 + e^{-100x}$$

Example3.

$$y' = -1000y + 3000 - 2000e^{-x}, \;\; y(0) = 0, \quad y(x) = 3 - 0.998e^{-1000x} - 2.002e^{-x}$$

Example 4

$$y' = -1000y + 999e^{-x}, \;\; y(0) = 1,$$

$$y(x) = e^{-x}$$

Example 1

gmrk order4	LobattoIIIA order4	gmrk order5	radIIA5	gmrk order 6	LobattoIIIA order6
0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00
2.40E-03	3.02E-03	1.81E-03	5.17E-04	1.45E-03	9.59E-04
5.74E-04	9.14E-04	3.26E-04	2.68E-05	2.10E-04	9.19E-05
1.37E-04	2.76E-04	5.90E-05	1.40E-06	3.03E-05	8.82E-06
3.29E-05	8.36E-05	1.07E-05	9.08E-08	4.39E-06	8.46E-07
7.87E-06	2.53E-05	1.92E-06	2.81E-08	6.36E-07	8.06E-08
1.90E-06	7.72E-06	3.48E-07	2.96E-08	9.21E-08	8.22E-09
4.41E-07	2.41E-06	6.23E-08	3.41E-08	1.34E-08	3.31E-10
1.18E-07	8.09E-07	1.20E-08	3.85E-08	1.90E-09	4.52E-10
1.73E-08	3.35E-07	1.39E-09	4.25E-08	3.06E-10	3.36E-10
1.36E-08	1.99E-07	1.07E-09	4.60E-08	1.82E-11	3.03E-10

ISSN 2229-5518

Example 2

gmrk order 4	LobattoIIIA order4	gmrk order 5	radIIA order 5	gmrk order 6	LobattoIIIA order6
0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00
2.40E-01	3.03E-01	1.81E-01	5.17E-02	1.45E-01	9.59E-02
5.74E-02	9.40E-02	3.26E-02	2.68E-03	2.10E-02	9.19E-03
1.37E-02	3.41E-02	5.90E-03	1.38E-04	3.03E-03	8.82E-04
3.29E-03	2.07E-02	1.07E-03	7.16E-06	4.39E-04	8.45E-05
7.88E-04	2.27E-02	1.93E-04	3.70E-07	6.36E-05	8.11E-06
1.89E-04	3.07E-02	3.48E-05	1.91E-08	9.21E-06	7.77E-07
4.52E-05	4.20E-02	6.29E-06	9.90E-10	1.33E-06	7.45E-08
1.08E-05	5.57E-02	1.14E-06	5.12E-11	1.93E-07	7.15E-09
2.59E-06	7.15E-02	2.05E-07	2.65E-12	2.80E-08	6.85E-10
6.21E-07	8.93E-02	3.71E-08	1.37E-13	4.05E-09	6.57E-11

Example 3

Gmrk order 4	LobattoIIIA order 4	gmrk order 5	radIIA5	Gmrk order 6	LobattoIIIA order6
0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00
3.24E-01	8.78E-01	2.43E-01	2.52E-02	1.95E-01	7.85E-01
1.05E-01	7.73E-01	5.93E-02	6.40E-04	3.80E-02	6.18E-01
3.41E-02	6.79E-01	1.44E-02	1.76E-05	7.41E-03	4.86E-01
1.10E-02	5.97E-01	3.52E-03	1.77E-06	1.45E-03	3.82E-01
3.58E-03	5.25E-01	8.56E-04	1.24E-06	2.81E-04	3.01E-01
1.16E-03	4.62E-01	2.10E-04	1.11E-06	5.62E-05	2.37E-01
3.76E-04	4.06E-01	4.99E-05	1.01E-06	9.75E-06	1.86E-01
1.23E-04	3.56E-01	1.33E-05	9.11E-07	3.00E-06	1.46E-01
3.89E-05	3.13E-01	2.21E-06	8.24E-07	4.05E-07	1.15E-01
1.36E-05	2.75E-01	1.47E-06	7.46E-07	8.16E-07	9.06E-02

Example 4

gmrk order4	LobattoIIIA order4	gmrk order5	radIIA5	gmrk order 6	LobattoIIIA order6
0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00
1.5E-09	3.49E-03	7.24E-11	1.08E-08	3.99E-12	1.41E-10
8.7E-10	6.26E-03	4.79E-11	1.00E-08	2.83E-12	1.66E-11
9.45E-10	8.41E-03	4.76E-11	9.10E-09	2.71E-12	1.02E-10
8.04E-10	1.00E-02	4.21E-11	8.23E-09	2.42E-12	2.39E-11
7.44E-10	1.13E-02	3.83E-11	7.45E-09	2.20E-12	7.55E-11
6.67E-10	1.21E-02	3.46E-11	6.74E-09	1.99E-12	2.59E-11
6.06E-10	1.27E-02	3.13E-11	6.10E-09	1.80E-12	5.68E-11
5.48E-10	1.30E-02	2.83E-11	5.52E-09	1.63E-12	2.52E-11
4.96E-10	1.31E-02	2.56E-11	4.99E-09	1.47E-12	4.34E-11
4.48E-10	1.30E-02	2.32E-11	4.52E-09	1.33E-12	2.31E-11

Conclusion

The numerical results from Tables 1 to 4 reveal the accuracy of the newly constructed class of block Generalized Adams implicit Runge –Kutta methods for step numbers 3, 4 and 5. It can be seen clearly from Table1, 3 and 4 that our new methods perform better than the Lobatto IIIA method of order 4, Lobatto IIIA of order 6 and the Radau IIA method of order 5 for the problems solved in examples 1,3 and 4. However, for example 2, Table 2 reveals that our new methods compete well with the Radau IIA method of order 5. It was also observed that the new methods have better stability regions than the conventional block generalized Adams methods.

References

[1] O.Axelson," A Class of A-Stable Methods". BIT, 9,185-199 (1969).

[2] J.C.Buctcher, "The Numerical Analysis of Ordinary Differential Equations", John Wiley and Sons, Chichester (1987)..

[3] J. C.Buctcher, G.Wanner," Runge – Kutta methods: Some Historical notes", Journal of Applied Numerical Mathematics 22, 113-115(1996).

[4] K. Burrage,J.C. Butcher, "Non Linear Stability for General Class of Differential Equations Method". BIT 20, 185 -203 (1980).

[5] F.H.Chipman, "A-stable Runge-Kutta Processes". BIT, 11, 384-366. (1971)

[6] J.P. Chollom, G.M. Kumleng, "A Class of Block Hybrid Adams Moulton Implicit Runge-Kutta (Bhamirk) Methods for Stiff Ordinary Differential Equations". *International Journal of Mathematics and Computations.* 14(1) 20-33 (2012).

[7] K. Dekker and J. G. Verwer, "Stability of Runge – Kutta methods for Stiff Nonlinear Differential Equations". *North-Holland, Amsterdam.(1984)*

[8] J.P.Chollom, J.N. Ndam and G.M.Kumleng, "On Some Properties of the Block Linear Multistep Methods". *Science World Journal,* 2(3), 11 – 17. (2007)

[9] S.Gonzalez-Pinto, S. perez-Rodriquez, J.I. Montijano,"On the numerical solution of stiff IVP's by Lobatto IIIA runge-kutta method". Journal of Computational and Applied Mathematics, 82,129-148 (1997).

[10] E. Hairer, and G. Wanner, "Solving Ordinary Differential Equations II", Springer-Verlag, Berlin (1996).

[11] G.M. Kumleng,"Continuous Generalized Adams Methods in Block Form for The Solution of Nonlinear Ordinary Differential Equations". PhD Thesis (Unpublished) University of Jos (2013)..

[12] I Lie and S. P. Norsett, ”Super convergence for Multistep Collocation”. Mathematics of Computation. 52(185), 65 – 79 (1989)..

[13] P. Onumanyi, D.O. Awoyemi, S.N. Jator and U.W. Sirisena,” New Linear Multistep Methods with Continuous Coefficients for First Order Initial Value Problems”. *Journal of the Nigerian Mathematical Society* .13, 37 – 51 (1994).

[14] D.G. Yakubu,” New Block Implicit Runge-Kutta Collocation Methods for Solving Stiff Differential Equations.” *Abacus* 31 ((2A) 119-129 (2004).

[15] D.G. Yakubu, N.H. Manjak, S.S. Buba, A.I. Maksha,”A Family of Uniformly Accurate Order Lobatto-Runge-Kutta Collocation Methods”. Comput. Appl. Math. 30(2) (2011).

- Geoffrey Micah Kumleng has obtained a PhD degree in Numerical analysis. Presently He lectures mathematics course in the University of Jos, Jos Nigeria, PH-+2348039687346. E-mail: kumleng_g@yahoo.com
- Joshua Pam Chollom has obtained a PhD degree in Numerical analysis. Presently He is an associate Professor fo mathematics in numerical analysis in the University of Jos, Jos Nigeria, PH-+2348035074503. E-mail: chollomjp@yahoo.com
- *Samson Omagwu is currently pursuing PhD degree program in mathematics (numerical analysis) in University of Jos, Jos Nigeria, PH-+2348028906636. E-mail: samsonomagwu@yahoo.com*